KATHOLIEKE UNIVERSITEIT
LEUVEN

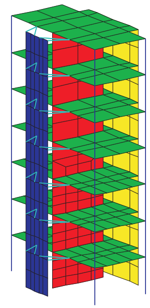
Joint workshop ULB-KUL on Structural Identification and Evaluation
17 February 2012

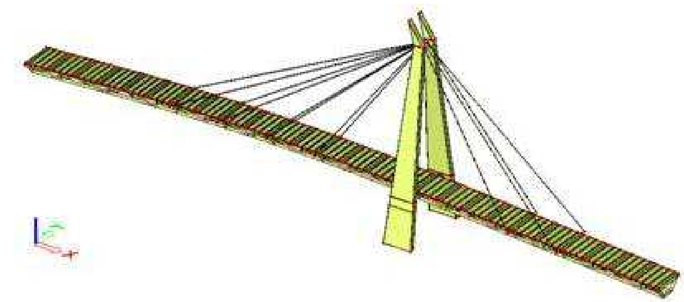
Uncertainty quantification in vibration-based parameter estimation

E. Simoen, G. Lombaert

ellen.simoen@bwk.kuleuven.be

Structural Mechanics Division
Department of Civil Engineering
KU Leuven

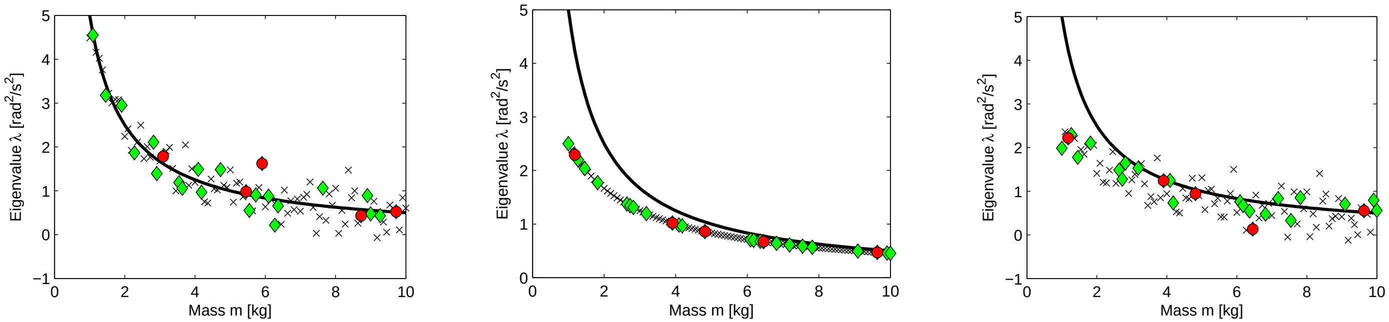


Vibration-based parameter estimation

- **Aim:** estimate parameters of a numerical model using observational data measured during laboratory or in situ experiments.
- **In structural dynamics:** finite element (FE) model updating. Calibration of FE model of a structure using time domain vibration data or modal features extracted from the data (e.g. eigenfrequencies, mode shapes, . . .).
- **Applications:** Structural Health Monitoring (SHM), improved predictions of structural response and reliability, ...

Uncertainty quantification

- Many uncertainties are associated with FE model updating:
  - Modeling uncertainty
  - Measurement uncertainty



Measurement error

- Accounting for uncertainties ensures more robust SHM and response predictions.
- Possible approaches:
  - Bayesian FE model updating
  - Fuzzy FE model updating
- **Application:** Damage assessment of a reinforced concrete beam
  - Deterministic FE model updating
  - Bayesian FE model updating
  - Resolution and uncertainty analysis

FE model updating

1. Extraction of modal characteristics $\tilde{\mathrm{d}} = \{\tilde{\boldsymbol{\lambda}}, \tilde{\boldsymbol{\phi}}\}^{\mathrm{T}}$ from vibration data

2. Construct a 2D or 3D finite element model to accurately represent the structural behaviour

- Parameterized by a number of (uncertain) parameters of interest $\boldsymbol{\theta}_{\mathrm{M}}$ (e.g. bending or torsional stiffness of elements or bearings, material properties, ...).
- Computation of modal characteristics

$$\mathbf{G}_{\mathrm{M}}(\boldsymbol{\theta}_{\mathrm{M}}) = \{\boldsymbol{\lambda}, \boldsymbol{\phi}\}^{T}$$

for a certain set of model parameters $\boldsymbol{\theta}_{\mathrm{M}}$ by solving the eigenvalue equation

$$\mathbf{K}(\boldsymbol{\theta}_{\mathrm{M}})\boldsymbol{\Phi} = \mathbf{M}\boldsymbol{\Phi}\boldsymbol{\Lambda}$$

3. Parameter estimation

- **Objective**: find the optimal parameters $\boldsymbol{\theta}_{\mathrm{M}}^{*}$ by minimizing a cost function that measures the discrepancy between measured and computed data:

$$\boldsymbol{\theta}_{\mathrm{M}}^{*} = \arg\min_{\boldsymbol{\theta}_{\mathrm{M}}} J\left(\mathbf{G}_{\mathrm{M}}(\boldsymbol{\theta}_{\mathrm{M}}), \tilde{\mathbf{d}}\right)$$

FE model updating

- **Objective**: find the optimal parameters $\boldsymbol{\theta}_M^*$ by minimizing a cost function that measures the discrepancy between measured and computed data:

$$\boldsymbol{\theta}_{\mathrm{M}}^* = \arg\min_{\boldsymbol{\theta}_{\mathrm{M}}} \; J\left(\mathbf{G}_{\mathrm{M}}(\boldsymbol{\theta}_{\mathrm{M}}), \tilde{\mathbf{d}}\right)$$

- Usually, a least squares cost function is used:

$$J\left(\mathbf{G}_{\mathrm{M}}(\boldsymbol{\theta}_{\mathrm{M}}), \tilde{\mathbf{d}}\right) = \parallel \mathbf{G}_{\mathrm{M}}(\boldsymbol{\theta}_{\mathrm{M}}) - \tilde{\mathbf{d}} \parallel^2 + \lambda \mathcal{P}(\boldsymbol{\theta}_{\mathrm{M}})$$

- Applied to the available vibration data (without regularization) this reduces to:

$$J(\boldsymbol{\theta}_{\mathrm{M}}, \tilde{\mathbf{d}}) = \sum_{r=1}^{N_m} \alpha_r \frac{(\lambda_r(\boldsymbol{\theta}_{\mathrm{M}}) - \tilde{\lambda}_r)^2}{\tilde{\lambda}_r^2} + \sum_{r=1}^{N_m} \beta_r \frac{\parallel \mathbf{L}\boldsymbol{\phi}_r(\boldsymbol{\theta}_{\mathrm{M}}) - \gamma_r \tilde{\boldsymbol{\phi}}_r \parallel^2}{\parallel \gamma_r \tilde{\boldsymbol{\phi}}_r \parallel^2}$$

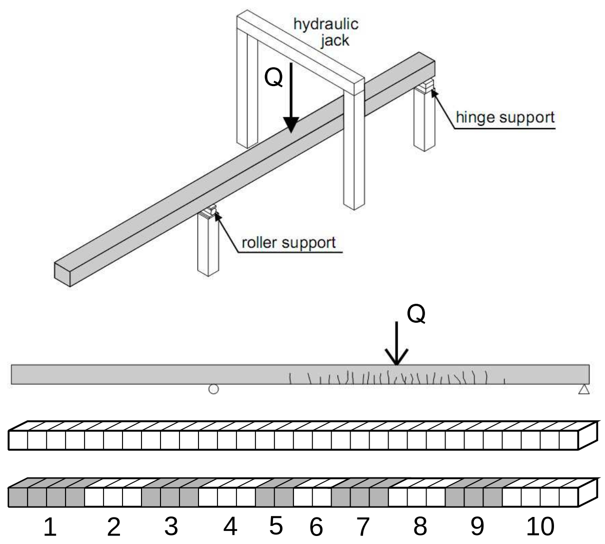
where

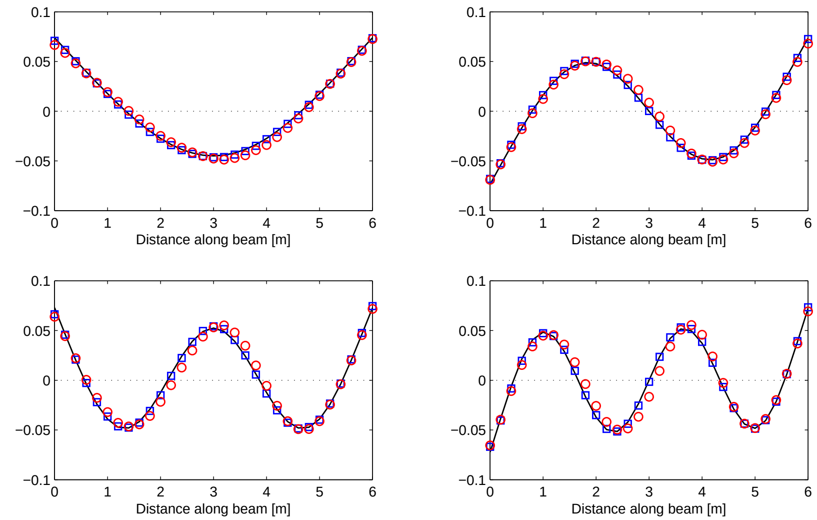
N_m	is the number of incorporated mode shapes
α_r and β_r	are weighting factors
$\mathbf{L} \in \mathbb{R}^{N_o \times N_d}$	is a binary matrix that selects the correct DOFs
γ_r	is a scaling factor

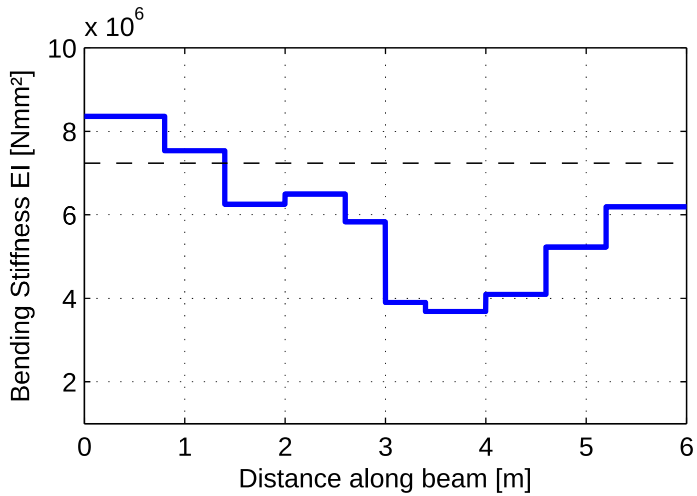
- Minimize objective function using suitable optimization algorithm.

FE model updating

- **Application**: reinforced concrete beam (6 m × 0.2 m × 0.25 m)



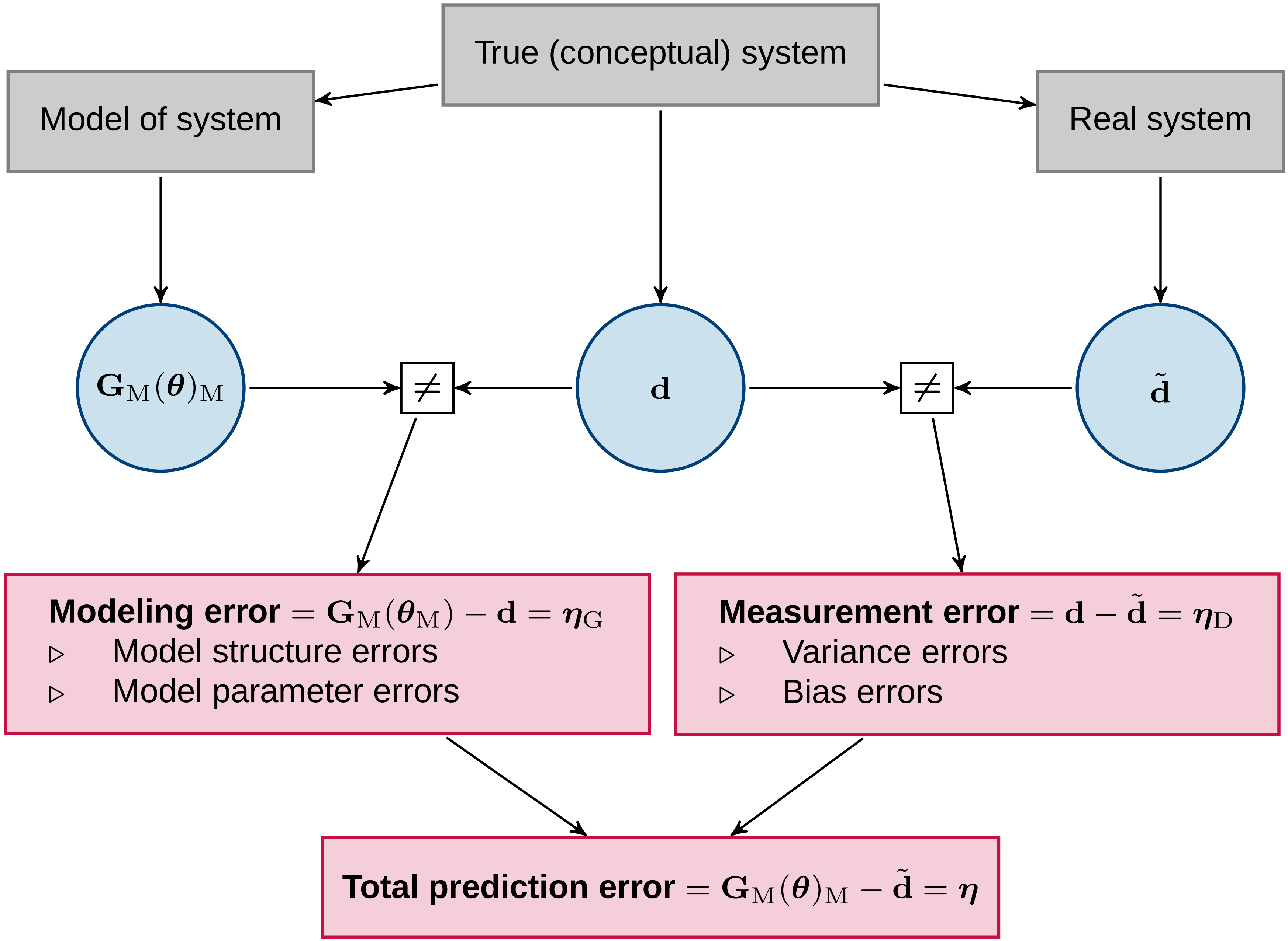




Mode	f_{exp} [Hz]	f_{init} [Hz]	Δf_{init} [%]
1	19.49	24.04	23.35
2	56.66	65.79	16.11
3	111.23	127.63	14.74
4	184.94	208.17	12.56
Mode		f_{upd} [Hz]	Δf_{upd} [%]
1	19.49	19.58	0.46
2	56.66	56.65	0.02
3	111.23	110.84	0.35
4	184.94	184.72	0.12

KATHOLIEKE UNIVERSITEIT LEUVEN

Uncertainty in FE model updating



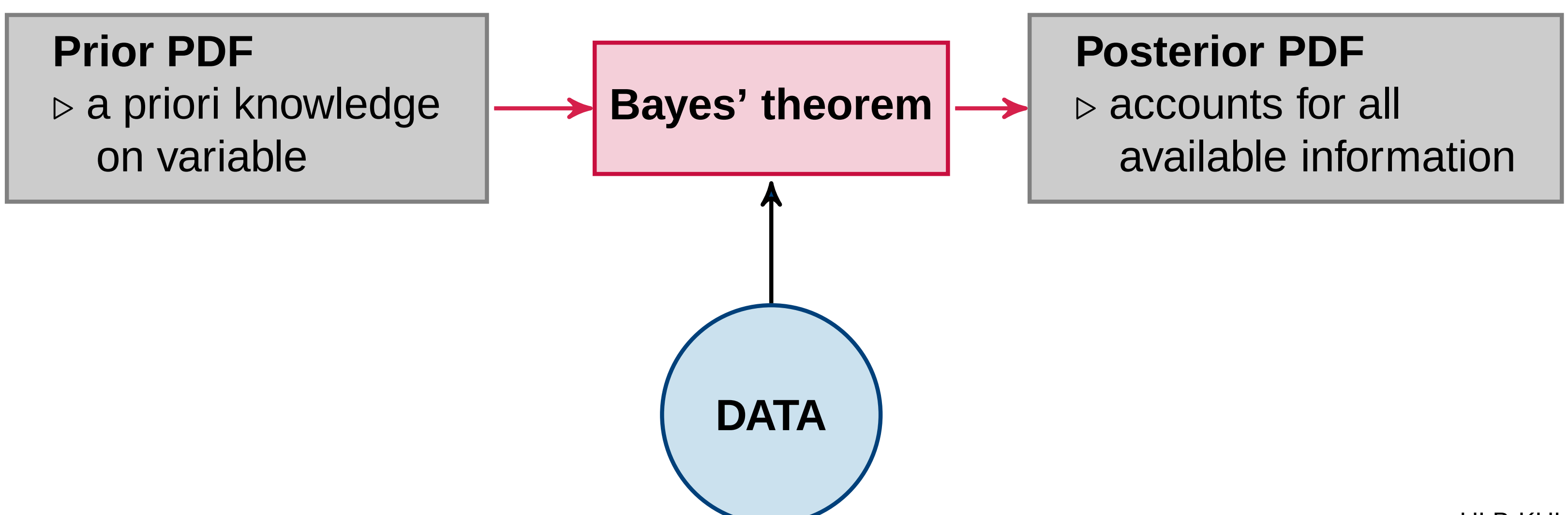
Bayesian FE model updating

Bayesian interpretation of probability

- Frequentist interpretation: probability = long term frequency of an event.
- Bayesian interpretation: probability = relative plausibility or degree of belief attributed to an event.

Bayesian FE model updating

- All uncertain parameters $\boldsymbol{\theta}$ are modeled as random variables.
- The errors are modeled as (known) random variables.
- The relative plausibility attributed to values of variables is reflected by the appointed probability density functions (PDFs).
- Using Bayes' theorem, a prior PDF is transformed into a posterior PDF



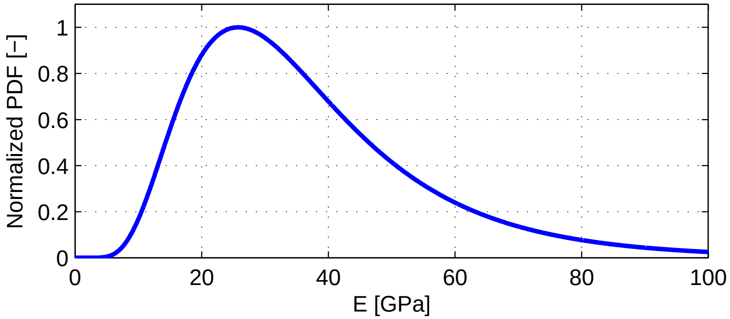
Reinforced concrete beam

Bayes' theorem

$$\boxed{\underbrace{p(\boldsymbol{\theta} \mid \tilde{\mathbf{d}})}_{\text{posterior PDF}} = c \underbrace{p(\boldsymbol{\theta})}_{\text{prior PDF}} \underbrace{p(\tilde{\mathbf{d}} \mid \boldsymbol{\theta})}_{\text{likelihood function}}}$$

Methodology

- Appoint a prior PDF $p(\boldsymbol{\theta})$ to the uncertain variables ($=$ FE model parameters)
  - For RC beam: $p(\theta_{\mathrm{M},i}) =$ lognormal distribution



- Construct likelihood function $p(\tilde{\mathbf{d}} \mid \boldsymbol{\theta}) \equiv L(\boldsymbol{\theta} \mid \tilde{\mathbf{d}})$
  - Total probability theorem:

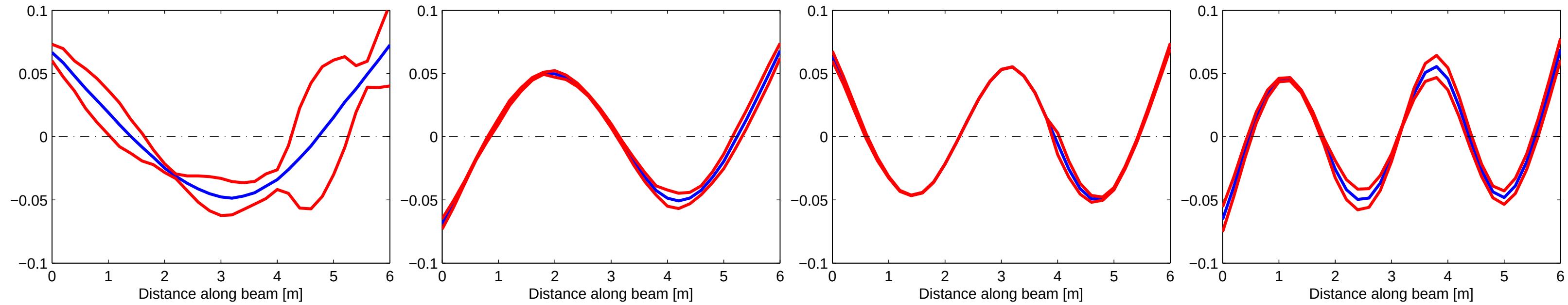
$$L(\boldsymbol{\theta} \mid \tilde{\mathbf{d}}) = \int_{D_{\mathcal{M}}} p(\boldsymbol{\eta}_{\mathrm{G}})\, p(\boldsymbol{\eta}_{\mathrm{D}})\, d\mathbf{d}$$

  - Probabilistic models for errors are needed

Reinforced concrete beam

Measurement error

- Estimation of covariance matrix $\mathbf{\Sigma}_\mathrm{D}$ characterizing a zero-mean Gaussian measurement error $\boldsymbol{\eta_D} \sim \mathcal{N}(\mathbf{0}, \mathbf{\Sigma}_\mathrm{D})$ [E. Reynders et al., MSSP, 2008]
- Estimated 50σ-intervals on mode shape displacements:



Modeling error

- Assumption of zero-mean Gaussian modeling error $\boldsymbol{\eta}_\mathrm{G} \sim \mathcal{N}(\mathbf{0}, \mathbf{\Sigma}_\mathrm{G})$
- Covariance matrix $\mathbf{\Sigma}_\mathrm{G}$ is a diagonal matrix with diagonal elements σ_i^2:

$$\begin{array}{lcl} \text{eigenvalues } \lambda_r & \rightarrow & \sigma_{\lambda,r} = \sigma_\lambda \tilde{\lambda}_r \\ \text{mode shapes } \boldsymbol{\phi}_r & \rightarrow & \sigma_{\phi,r} = \sigma_\phi \parallel \tilde{\boldsymbol{\phi}}_r \parallel \end{array}$$

where $\sigma_\lambda = \sigma_\phi = 0.01$

$\Rightarrow$ **Likelihood function** $L(\boldsymbol{\theta} \mid \tilde{\mathrm{d}}) \sim \mathcal{N}(\mathbf{0}, \mathbf{\Sigma}_\mathrm{D} + \mathbf{\Sigma}_\mathrm{G}) \sim \mathcal{N}(\mathbf{0}, \mathbf{\Sigma}_\eta)$

Reinforced concrete beam

Maximum A Posteriori (MAP) estimate

- Posterior PDF for RC beam:

$$
\begin{aligned}
p(\boldsymbol{\theta} \mid \tilde{\mathbf{d}}) &\propto L(\boldsymbol{\theta} \mid \tilde{\mathbf{d}})\, p(\boldsymbol{\theta}) \\
&\propto \exp\left[-\frac{1}{2}\left(\mathbf{G}_{\mathrm{M}}(\boldsymbol{\theta}_{\mathrm{M}}) - \tilde{\mathbf{d}}\right)^T \boldsymbol{\Sigma}_\eta^{-1} \left(\mathbf{G}_{\mathrm{M}}(\boldsymbol{\theta}_{\mathrm{M}}) - \tilde{\mathbf{d}}\right)\right] \\
&\quad \times \prod_{i=1}^{N} \frac{1}{\theta_{\mathrm{M},i}} \exp\left[-\frac{1}{2\sigma_i^2}(\log \theta_{\mathrm{M},i} - \mu_i)^2\right]
\end{aligned}
$$

- Maximum A Posteriori estimate:

$$
\begin{aligned}
\hat{\boldsymbol{\theta}}_{\mathrm{M}}^{\mathrm{MAP}} &= \arg\max_{\boldsymbol{\theta}} \left\{p(\boldsymbol{\theta} \mid \tilde{\mathbf{d}})\right\} = \arg\min_{\boldsymbol{\theta}} \left\{-\log p(\boldsymbol{\theta} \mid \tilde{\mathbf{d}})\right\} \\
&= \arg\min_{\boldsymbol{\theta}} \left\{\frac{1}{2}\left(\mathbf{G}_{\mathrm{M}}(\boldsymbol{\theta}_{\mathrm{M}}) - \tilde{\mathbf{d}}\right)^T \boldsymbol{\Sigma}_\eta^{-1} \left(\mathbf{G}_{\mathrm{M}}(\boldsymbol{\theta}_{\mathrm{M}}) - \tilde{\mathbf{d}}\right)\right. \\
&\quad \left. + \sum_{i=1}^{N} \left[\frac{1}{2\sigma_i^2}(\log \theta_{\mathrm{M},i} - \mu_i)^2 + \theta_{\mathrm{M},i}\right]\right\}
\end{aligned}
$$

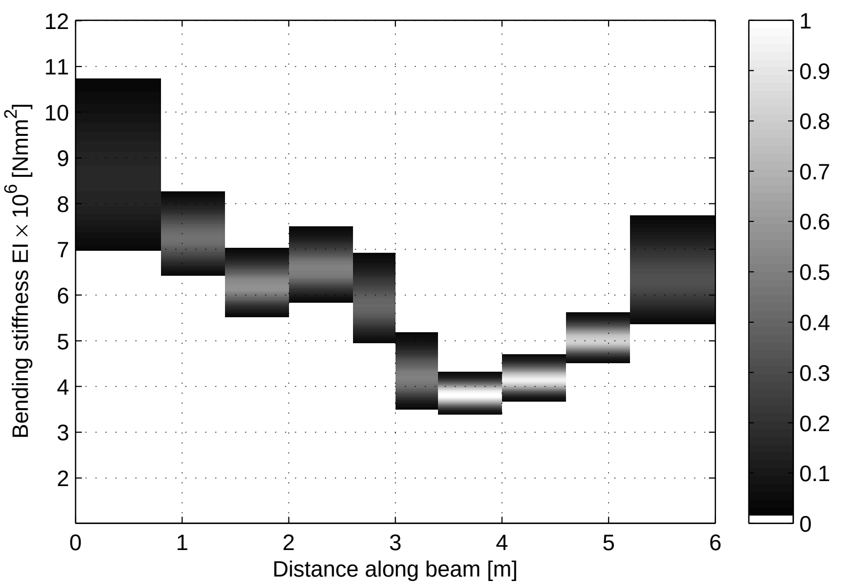
- Note that this objective function corresponds to a generalized least squares objective function **with regularization**.
- For a diagonal prediction error covariance matrix, the weights appointed to the discrepancies are inversely proportional to the attributed variances; i.e. more weight is given to more accurate data.

Reinforced concrete beam

Computation of posterior joint PDF and marginal PDFs

- Often challenging (high-dimensional integrals, complex models, . . .)
  - Analytical when possible
  - Markov Chain Monte Carlo sampling (MCMC) – increases computational cost
  - Asymptotic approximations [C. Papadimitriou et al., JEM, 1997]

Results for RC beam: posterior PDFs – 99% confidence bounds



Sub-structure	E [GPa] – Dam. state		
	MAP	$\mu_{po}(\sigma_{po})$	c_{po} [%]
1	42.76	46.56 (6.74)	14.48
2	38.10	37.78 (2.16)	5.73
3	31.88	32.27 (1.67)	5.17
4	34.76	34.25 (1.88)	5.50
5	30.14	30.45 (2.46)	8.07
6	21.58	22.12 (1.94)	8.77
7	19.67	19.81 (0.94)	4.76
8	21.47	21.54 (1.02)	4.74
9	26.29	26.06 (1.16)	4.46
10	31.93	33.61 (3.03)	9.03

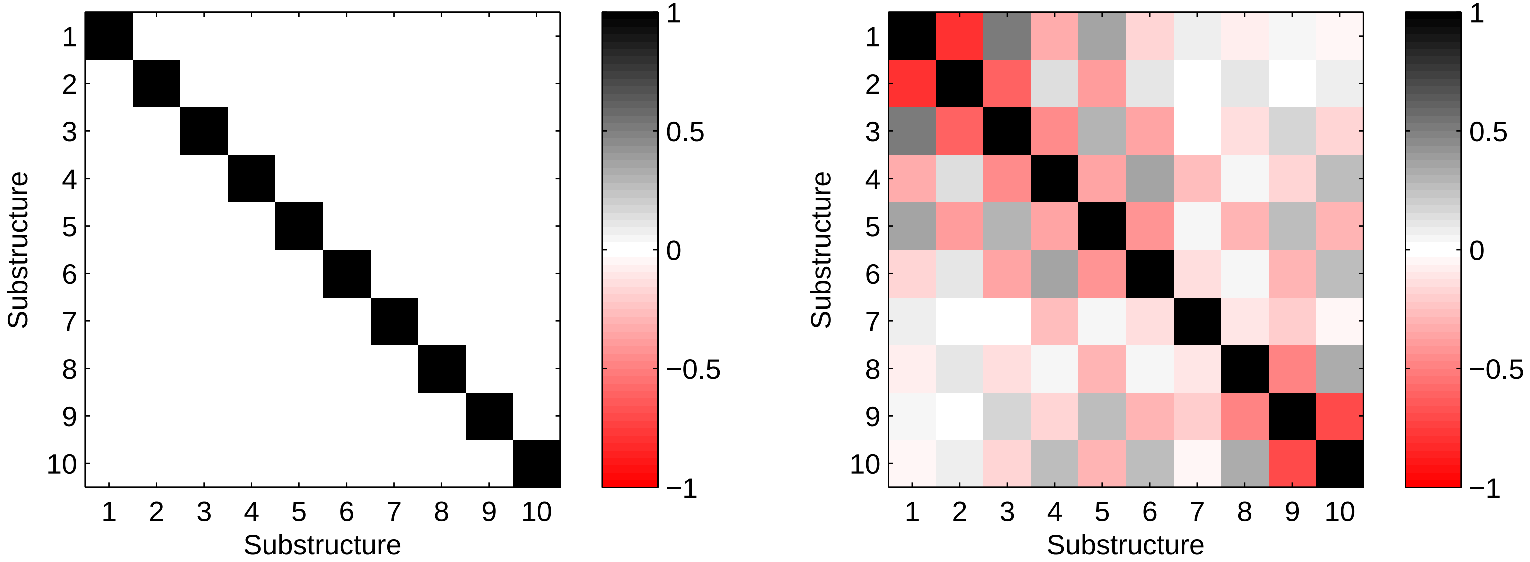
Resolution analysis

Analysis of resulting uncertainty and resolution

- Posterior mean values and standard deviations
- Prior and posterior covariance matrices $\mathbf{S}_{pr}$ and $\mathbf{S}_{po}$
- Prior and posterior correlation coefficient matrices

$$[\mathbf{C}_{pr}]_{ij} = \frac{[\mathbf{S}_{pr}]_{ij}}{\sqrt{[\mathbf{S}_{pr}]_{ii}[\mathbf{S}_{pr}]_{jj}}} \qquad [\mathbf{C}_{po}]_{ij} = \frac{[\mathbf{S}_{po}]_{ij}}{\sqrt{[\mathbf{S}_{po}]_{ii}[\mathbf{S}_{po}]_{jj}}}$$

- Prior and posterior correlation coefficient matrix for the **RC beam**



Resolution analysis

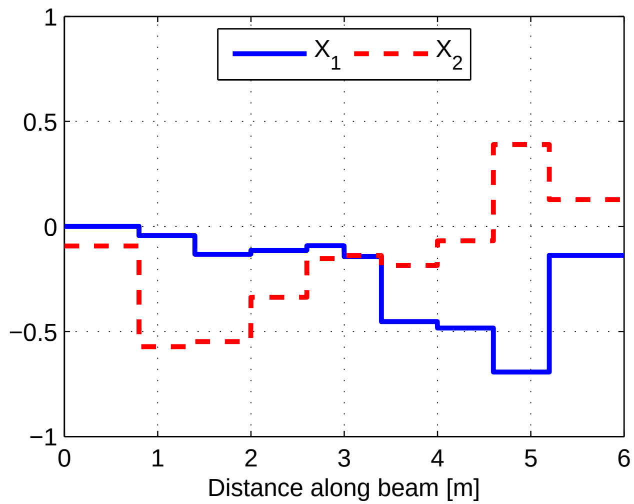
Eigenvalue analysis [A. Tarantola, SIAM, 2005],[A. Duijndam, GP, 1988]

- Solve the eigenvalue problem

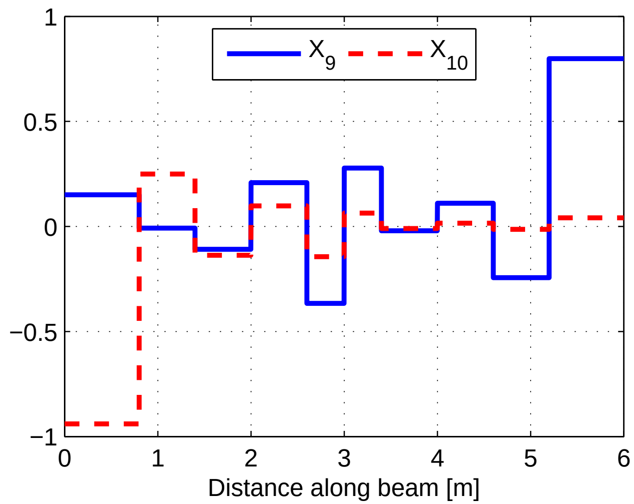
$$\boxed{\mathbf{S}_{po}\mathbf{X} = \mathbf{\Lambda}\mathbf{S}_{pr}\mathbf{X}}$$

  - Rotate coordinate system to orthogonal (uncorrelated) directions $\mathbf{X}$, ranked from highest to lowest reduction from prior to posterior variance.
  - The directions $\mathbf{X}$ in the parameter space correspond to linear combinations of parameters, ranked according to decreasing resolution.

- Best and worst resolved linear parameter combinations for the **RC beam**:



Best resolved



Worst resolved

Relation to information entropy [E. Simoen et al., EURODYN2011, 2011]

- Relative entropy or Kullback-Leibler divergence D_{KL} between prior and posterior PDF:

$$D_{KL} = \int p(\boldsymbol{\theta}|\mathbf{d}) \log \left(\frac{p(\boldsymbol{\theta}|\mathbf{d})}{p(\boldsymbol{\theta})} \right) d\boldsymbol{\theta}$$

$$\approx -\frac{1}{2} \sum_{k=1}^{N} \log \lambda_k$$

  where λ_k are the eigenvalues of the eigenvalue problem $\mathbf{S}_{po}\mathbf{X} = \boldsymbol{\Lambda}\mathbf{S}_{pr}\mathbf{X}$

- Value $-\log \lambda_k$ reflects the contribution of the corresponding eigenvector $\mathbf{X}_k$ to the total entropy reduction due to the data

- **RC beam**
  - $D_{KL} = 25.48 \Rightarrow$ average reduction of uncertainty ≈ 2.2 orders of magnitude
  - $-\log \lambda_1 = 7.41 \approx 3.2$ orders of magnitude reduction in direction defined by $\mathbf{X}_1$
  - $-\log \lambda_{10} = 2.09 \approx 0.9$ orders of magnitude reduction in direction defined by $\mathbf{X}_{10}$

KATHOLIEKE UNIVERSITEIT LEUVEN

Conclusions

- Accounting for uncertainty in FE model updating by applying a probabilistic Bayesian approach provides more robust FE model updating results and subsequent response predictions.
- The Bayesian approach proves very effective in estimating the resulting uncertainty on the model parameters. Furthermore, it is shown that making use of prior information effectively corresponds to regularization of the corresponding deterministic problem, while naturally enforcing constraints included in the available prior information (e.g. positivity of the parameters, . . .).
- Because the method is established in a probabilistic framework, extensive post-processing of the results is possible, i.e. computation of mean and MAP values, standard deviations, correlation coefficient matrices, . . .
- An eigenvalue analysis based on the prior and posterior covariance matrices complements these methods effectively, and provides a link to information entropy.
- The comprehensive resolution analysis allows for a detailed insight into the resulting uncertainties and their underlying causes.